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On largeness and multiplicity of the first eigenvalue of finite area hyperbolic surfaces

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Abstract We apply topological methods to study the smallest non-zero number λ_1 in the spectrum of the Laplacian on finite area hyperbolic surfaces. For closed hyperbolic surfaces of genus two we show that the set $\{S \in \mathcal{M}_2 : \lambda_1(S) > \frac{1}{4}\}$ is unbounded and disconnects the moduli space $\mathcal{M}_2$. Using this, for genus $g \geq 3$, we show the existence of eigenbranches that start as λ_1 and eventually becomes $>\frac{1}{4}$.

Keywords Hyperbolic surfaces · Laplace operator · First eigenvalue · Small eigenvalues

1 Introduction

In this paper we identify hyperbolic surfaces with quotients of the Poincaré upper halfplane $\mathbb{H}$ by discrete torsion free subgroups of $\mathrm{PSL}(2, \mathbb{R})$ called *Fuchsian groups*. The *Laplacian* on $\mathbb{H}$ is the differential operator Δ which associates to a real-valued C^2-function f the function

$$\Delta f(z) = -y^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right). \tag{1.1}$$

For any Fuchsian group Γ, the induced differential operator on $S = \mathbb{H}/\Gamma$, $\Delta = \Delta_S$ is called the Laplacian on S. It is a non-negative operator whose spectrum $\mathrm{spec}(\Delta)$ is contained in a smallest interval $[\lambda_0(S), \infty) \subset \mathbb{R}^+ \cup \{0\}$ with $\lambda_0(S) \geq 0$. Points in the discrete spectrum will be referred to as *eigenvalues*. In particular this means $\lambda \geq 0$ is an eigenvalue if there exists a non-zero C^2-function $f \in L^2(S)$, called a λ-*eigenfunction*, such that $\Delta f = \lambda f$. The pair (λ, f) is called an *eigenpair*. When $0 < \lambda \leq 1/4$, λ is called a *small eigenvalue*, f is called a *small eigenfunction* and the pair (λ, f) is called a *small eigenpair*. Recall that we consider only real-valued functions and so any eigenfunction is a real-valued function.

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We shall restrict ourselves to hyperbolic surfaces with finite area. Any such surface S is homeomorphic to a closed Riemann surface $\overline{S}$ of certain genus g from which some n many points are removed. In that case S is called a finite area hyperbolic surface of type (g, n). Each of these n points is called a *puncture* of S.

The spectrum of the Laplacian of a closed hyperbolic surface S consists of a discrete set:

$$0 = \lambda_0 < \lambda_1(S) \leq \cdots \leq \lambda_n(S) \leq \cdots \infty \tag{1.2}$$

such that $\lambda_i(S) \to \infty$ as $i \to \infty$. Each number in the above sequence is repeated according to its multiplicity as eigenvalue. The number $\lambda_i(S)$ is called the i-th eigenvalue of S. The moduli space of genus g closed hyperbolic surfaces is denoted by $\mathcal{M}_g$. It is known that the map $\lambda_i : \mathcal{M}_g \to \mathbb{R}$ that assigns a surface $S \in \mathcal{M}_g$ to its i-th eigenvalue $\lambda_i(S)$ is continuous and bounded [4]. Hence

$$\Lambda_i(g) = \sup_{S \in \mathcal{M}_g} \lambda_i(S) < \infty. \tag{1.3}$$

For non-compact hyperbolic surfaces of finite area the spectrum of the Laplacian is more complicated. It consists of both continuous and discrete components (see [14] for detail). However, the part of the spectrum lying in $[0, \frac{1}{4})$ is discrete. Keeping resemblance to the above definition, for any hyperbolic surface S, let us define $\lambda_1(S)$ to be the smallest positive number in spec(Δ). In particular, if $\lambda_1 < \frac{1}{4}$ then it is an eigenvalue. The function λ_1, so defined, is bounded by $\frac{1}{4}$ because S has a continuous spectrum on $[\frac{1}{4}, \infty)$. As before we consider the moduli space $\mathcal{M}_{g,n}$ of finite area hyperbolic surfaces of type (g, n) and define

$$\Lambda_1(g, n) = \sup_{S \in \mathcal{M}_{g,n}} \lambda_1(S). \tag{1.4}$$

In [22] Atle Selberg proved that for any congruence subgroup Γ of SL$(2, \mathbb{Z})$

$$\lambda_1(\mathbb{H}/\Gamma) \geq \frac{3}{16}. \tag{1.5}$$

Recall that a congruence subgroup is a discrete subgroup of SL$(2, \mathbb{Z})$ that contains one of the Γ_n where

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : a \equiv 1 \equiv d \ \text{ and } \ b \equiv 0 \equiv c \ (\mathrm{mod}\ n) \right\} \tag{1.6}$$

is the principal congruence subgroup of level n. Moreover he conjectured:

Conjecture 1.1 *For any congruence subgroup Γ, $\lambda_1(\mathbb{H}/\Gamma) \geq \frac{1}{4}$.*

Huxley [13] proved this conjecture for Γ_n with $n \leq 6$. Several attempts have been made to prove it (see [14, Chapter 11] for details) in the general case. The best known bound is $\frac{975}{4096}$ due to Kim and Sarnak [18]. This conjecture motivated, in particular, the question of our interest:

Question 1.1 *Given any genus $g \geq 2$ does there exist a closed hyperbolic surface of genus g with λ_1 at least $\frac{1}{4}$?*

A slightly weaker question than the above one would be: Is $\Lambda_1(g) \geq \frac{1}{4}$? This question is studied in [5] by Buser et al. and in [6] by Brooks and Makover. The ideas in [5,6], in the light of the bound of Kim and Sarnak [18], provide the following.

Theorem 1.1 *Given any* $\epsilon > 0$*, there exists* $N_\epsilon \in \mathbb{N}$ *such that for any* $g \geq N_\epsilon$ *there exist closed hyperbolic surfaces of genus* g *with* $\lambda_1 \geq \frac{975}{4096} - \epsilon$.

The constant $\frac{975}{4096}$ in the above theorem can be replaced by $\frac{1}{4}$ if Conjecture 1.1 is true. Hence it is tempting to conjecture:

Conjecture 1.2 *For every* $g \geq 2$ *there exists a closed hyperbolic surface of genus* g *whose* λ_1 *is at least* $\frac{1}{4}$.

Remark 1.1 Observe that even if Selberg's conjecture (Conjecture 1.1) is true, Theorem 1.1 would not provide a positive answer to Conjecture 1.2. However, it would imply that for special values of g (see [5]) $\Lambda_1(g) \geq \frac{1}{4}$ and $\liminf_{g \to \infty} \Lambda_1(g) \geq \frac{1}{4}$.

The existence of genus two hyperbolic surfaces with $\lambda_1 > \frac{1}{4}$ has been known in the literature for sometime [15]. It is known that the *Bolza* surface has λ_1 approximately 3.8 (see [23] for more details). We consider the subset $\mathcal{B}_2(\frac{1}{4}) = \{S \in \mathcal{M}_2 : \lambda_1(S) > \frac{1}{4}\}$ of the moduli space $\mathcal{M}_2$. From the continuity of λ_1 it is clear that $\mathcal{B}_2(\frac{1}{4})$ is open. Our first result, in some sense, describes how large the open subset $\mathcal{B}_2(\frac{1}{4})$ is.

Theorem 1.2 $\mathcal{B}_2\left(\frac{1}{4}\right)$ *is an unbounded set that disconnects* $\mathcal{M}_2$.

Sketch of the proof of Theorem 1.2 First we prove that $\mathcal{B}_2(\frac{1}{4})$ disconnects $\mathcal{M}_2$. For that we argue by contradiction and assume that $\mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ is connected. Now for any $S \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$, $\lambda_1(S)$ is small and hence has multiplicity exactly one by Proposition 1.1. In particular, the space of $\lambda_1(S)$-eigenfunctions is one dimensional and so the *nodal set* $\mathcal{Z}_S$ of $\lambda_1(S)$-eigenfunctions is defined without any ambiguity (see Sect. 2.2). We shall see that under our assumptions $\mathcal{Z}_S$ is a disjoint union of simple closed curves. With the help of this property we shall deduce that $\mathcal{Z}_S$ is constant, up to isotopy, on $\mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$. Finally, using an argument involving geodesic pinching (Proposition 3.1) we shall show that there exist surfaces S_1 and S_2 in $\mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ such that $\mathcal{Z}_{S_1}$ is not isotopic to $\mathcal{Z}_{S_2}$. This provides the desired contradiction. The proof of the rest of the theorem uses similar topological arguments.

For finite area hyperbolic surfaces with Euler characteristic -2 the ideas in the above proof carries over to provide the following.

Theorem 1.3 *For any* (g, n) *with* $2g - 2 + n = 2$ *(i.e.* $(g, n) = (2, 0), (1, 2)$ *or* $(0, 4)$*) the set* $\mathcal{C}_{g,n}(\frac{1}{4}) = \{S \in \mathcal{M}_{g,n} : \lambda_1(S) \geq \frac{1}{4}\}$ *disconnects* $\mathcal{M}_{g,n}$*. Moreover for* $(g, n) = (2, 0)$ *and* $(1, 2)$ *it is unbounded.*

1.1 Eigenvalue branches

Recall that the moduli space $\mathcal{M}_g$ is the quotient of the Teichmüller space $\mathcal{T}_g$ by the *Teichmüller modular group* M_g (see [4]). We are shifting from the moduli space to the Teichmüller space mainly because we wish to talk about analytic paths which involve coordinates and on $\mathcal{T}_g$ one has the Fenchel–Nielsen coordinates (given a pants decomposition) which is easy to describe.

Let $\gamma : [0, 1] \to \mathcal{T}_2$ be an analytic path. Since, in this case, λ_1 is simple as long as small (by Proposition 1.1), the function $\lambda_1(S^t)$ ($S^t = \gamma(t)$) is also analytic (see Theorem 1.4) if $\lambda_1(S^t) \leq \frac{1}{4}$ for all $t \in [0, 1]$. For higher genus λ_1 may not be simple even if small (see Sect. 1.2). Therefore, for an analytic path $\gamma : [0, 1] \to \mathcal{T}_g$, $\lambda_1(S^t)$ is continuous but need not be analytic even if $\lambda_1(S^t) \leq \frac{1}{4}$ for all $t \in [0, 1]$. However we have the following result from [4, Theorem 14.9.3]:

Theorem 1.4 *Let $(S^t)_{t\in I}$ be a real analytic path in $\mathcal{T}_g$. Then there exist real analytic functions $\lambda_k^t : I \to \mathbb{R}$ such that for each $t \in I$ the sequence (λ_k^t) consist of all eigenvalues of S^t (listed with multiplicities, though not in increasing order).*

Each function λ_k^t is called a branch of eigenvalues along S^t. More precisely

Definition 1.1 Let $\alpha : [0, 1] \to \mathcal{T}_g$ be an analytic path. An analytic function $\lambda_t : [0, 1] \to \mathbb{R}$ is called a branch of an eigenvalue along α if, for each t, λ_t is an eigenvalue of $\alpha(t)$. If $\lambda_0 = \lambda_i(\alpha(0))$ then we shall say that λ_t is a branch of eigenvalues along α that starts as λ_i. If the underlying path α is fixed then we shall skip referring to it.

Here, instead of considering λ_1, we consider branches of eigenvalues that start as λ_1 and modify question 1.1 as:

Question 1.2 *For any $g \geq 2$ does there exist branches of eigenvalues in $\mathcal{T}_g$ that start as λ_1 and exceeds $\frac{1}{4}$ eventually ?*

Fortunately this modified question turns out to be much easier than the original one and we have a positive answer to it.

Theorem 1.5 *For any $g \geq 2$ there are branches of eigenvalues in $\mathcal{T}_g$ that start as λ_1 and take values strictly bigger than $\frac{1}{4}$.*

Recall that $\mathcal{T}_2$ can be embedded in $\mathcal{T}_g$ as an analytic subset containing surfaces with certain symmetries (see Sect. 4). The branches in Theorem 1.5 will be obtained by composing the branches in $\mathcal{T}_2$ by the above embedding $\Pi : \mathcal{T}_2 \to \mathcal{T}_g$. We shall use a *geodesic pinching* argument to prove that among these branches there are ones that start as λ_1.

1.2 Multiplicity

For any eigenvalue λ of S, the dimension of $\ker(\Delta - \lambda.1)$ is called the multiplicity of λ. If the multiplicity of λ_1 were one for all closed hyperbolic surfaces of genus g then Theorem 1.5 would have showed the existence of surfaces with $\lambda_1 > \frac{1}{4}$ implying Conjecture 1.2. However this is not the case and in fact the following is proved in [10]:

Theorem 1.6 *For every $g \geq 3$ and $n \geq 0$ there exists a surface $S \in \mathcal{M}_{g,n}$ such that $\lambda_1(S)$ is small and has multiplicity equal to the integral part of $\frac{1+\sqrt{8g+1}}{2}$.*

For $g \geq 3$ the above bound is more than 3. Hence our methods in Theorem 1.2 for $g = 2$ do not work for $g \geq 3$. In [20] the following upper bound on the multiplicity of a small eigenvalue is proved.

Proposition 1.1 *Let S be a finite area hyperbolic surface of type (g, n). Then the multiplicity of a small eigenvalue of S is at most $2g - 3 + n$.*

Our last result is an improvement of this result for finite area hyperbolic surfaces of type $(0, n)$. Recall that for any finite area hyperbolic surface if $\frac{1}{4}$ is an eigenvalue then it must be a *cuspidal* eigenvalue (see Sect. 2.3). Now, hyperbolic surfaces of type $(0, n)$ can not have small cuspidal eigenvalues by [20, Proposition 2] (see also [13]). Therefore, **for a finite area hyperbolic surfaces S of type $(0, n)$ if $\lambda_1(S)$ is a small eigenvalue then automatically $\lambda_1(S) < \frac{1}{4}$.**

Theorem 1.7 *Let S be a finite area hyperbolic surface of genus 0. If $\lambda_1(S)$ is a small eigenvalue then the multiplicity of $\lambda_1(S)$ is at most three.*

Sketch of proof Let $\overline{S}$ denote the closed surface obtained by filling in the punctures of S. By assumption $\lambda_1(S)$ is small. Following the discussion above $\lambda_1(S) < \frac{1}{4}$. Let ϕ be a $\lambda_1(S)$-eigenfunction with nodal set $\mathcal{Z}(\phi)$. Let $\overline{\mathcal{Z}(\phi)}$ be the closure of $\mathcal{Z}(\phi)$ in $\overline{S}$ which is a finite graph by Lemma 2.1.

Using **Jordan curve theorem** and **Courant's nodal domain theorem** (see Sect. 2.2) we shall deduce the simple description of $\overline{\mathcal{Z}(\phi)}$ as a simple closed curve in $\overline{S}$. In particular, if one of the punctures p of S lies on $\overline{\mathcal{Z}(\phi)}$ then the number of arcs in $\overline{\mathcal{Z}(\phi)}$ emanating from p is at most two.

Let p be one of the punctures of S. We shall deduce that in any cusp around p any $\lambda_1(S)$-eigenfunction ϕ has a Fourier development of the form:

$$\phi(x, y) = \phi_0 y^{1-s} + \sum_{j\geq 1}\sqrt{\frac{2jy}{\pi}}K_{s-\frac{1}{2}}(jy)(\phi_j^e \cos(j.x) + \phi_j^o \sin(j.x)) \tag{1.7}$$

where $\lambda_1(S) = s(1-s)$ with $s \in (\frac{1}{2}, 1]$ and K is the modified Bessel function of exponential decay (see Sect. 2.3). Denote the vector space generated by $\lambda_1(S)$-eigenfunctions by $\mathcal{E}_1$ and consider the map $\pi : \mathcal{E}_1 \to \mathbb{R}n^3$ given by $\pi(\phi) = (\phi_0, \phi_1^e, \phi_1^o)$. This is a linear map and so if $\dim \mathcal{E}_1 > 3$ then $\ker \pi$ is non-empty. Let $\psi \in \ker \pi$ i.e. $\psi_0 = \psi_1^e = \psi_1^o = 0$. Then by the result [17] of Judge, the number of arcs in $\overline{\mathcal{Z}(\psi)}$ emanating from p is at least four, a contradiction to the above description of $\overline{\mathcal{Z}(\phi)}$ at p.

2 Preliminaries

In this section we recall some definitions and results that will be necessary in the later sections. We begin by some backgrounds from topology where we recall a particular form of the Euler–Poincaré formula. Then we recall some results on the structure of nodal sets of eigenfunctions. The last part recalls the Fourier expansion of cusp forms in a cusp.

2.1 Backgrounds from topology

Here we recall some background materials from topology. A (finite) graph G on S consists of a pair (V, E) where V, called the set of *vertices* of G, is a finite collection of points of S and E, called the set of *edges* of G, is a finite collection of mutually non-intersecting embedded arcs in S joining the points in V. If an edge e joins two vertices v and w then we say that e is *adjacent* to v and w. The total number of edges adjacent to a vertex is called the *degree* of the vertex. A vertex is called an *isolated vertex* if its degree is zero and a *free vertex* if its degree is one. It is not very difficult to observe that the Euler characteristic of a finite graph without any isolated or free vertex is always ≤ 0.

Let $G = (V, E)$ be a graph on S. Since both V and E are finite it is easy to observe that for any $\epsilon > 0$ small enough the ϵ-neighborhood $N_\epsilon(G)$ of G has piecewise smooth boundary and deformation retracts to G. Moreover, any component C of $S\backslash N_\epsilon(G)$ is a deformation retraction of the unique component C' of $S\backslash G$ that contains C. Now choose two such constants ϵ, δ with $\delta < \epsilon$ and consider the decomposition of S into the components of $S\backslash N_\delta(G)$ and $N_\epsilon(G)$. Then one can use the Mayer–Vietoris sequence [11, p-149] to observe that

$$\chi(S) = \sum_i \chi(D_i) + \chi(N_\epsilon(G)) \tag{2.1}$$

where D_i runs over the components of $S\backslash N_\delta(G)$ and $\chi(A)$ denotes the Euler characteristic of A. Since $N_\epsilon(G)$ deformation retracts to G and each component C of $S\backslash N_\delta(G)$ is a deformation retraction of the unique component C' of $S\backslash G$ that contains C we obtain

$$\chi(S) = \sum_i \chi(D_i) + \chi(G) \tag{2.2}$$

where D_i runs over the components of $S\backslash G$. This formula is sometimes called the **Euler–Poincaré formula**.

2.2 Nodal sets

For any function $f : S \to \mathbb{R}$, the set $\{x \in S : f(x) = 0\}$ is called the *nodal set* $\mathcal{Z}(f)$ of f. Observe that $\mathcal{Z}(f)$ is invariant under multiplication by non-zero constants i.e. $\mathcal{Z}(f) = \mathcal{Z}(c.f)$ for any $c \neq 0$. Each component of $S\backslash\mathcal{Z}(f)$ is called a *nodal domain* of f. In a neighborhood of a regular point $p \in \mathcal{Z}(f)$ ($\nabla_p f \neq 0$) the implicit function theorem implies that $\mathcal{Z}(f)$ is a smooth curve. In a neighborhood of a critical point $p \in \mathcal{Z}(f)$ ($\nabla_p f = 0$), it is not so simple to describe $\mathcal{Z}(f)$. When f is an eigenfunction of the Laplacian we have the following description due to Cheng [8]:

Theorem 2.1 *Let S be a surface with a C^∞ metric. Then, for any solution of the equation $(\Delta - h)\phi = 0$, $h \in C^\infty(S)$, one has:*

(i) *Critical points on the nodal set $\mathcal{Z}(\phi)$ are isolated.*
(ii) *Any critical point in $\mathcal{Z}(\phi)$ has a neighborhood N in S which is diffeomorphic to the disc $\{z \in \mathbb{C} : |z| < 1\}$ by a C^1-diffeomorphism that sends $\mathcal{Z}(\phi) \cap N$ to an equiangular system of rays.*

Remark 2.1 (1) $\mathcal{Z}(\phi)$ does not contain any isolated or free vertex.
(2) If $p \in \mathcal{Z}(\phi)$ is a critical point of ϕ then the degree of the graph $\mathcal{Z}(\phi)$ at p is at least 4. Hence if a component of $\mathcal{Z}(\phi)$ is a simple closed loop then it is automatically smooth.

When S is closed Theorem 2.1 implies that $\mathcal{Z}(\phi)$ is a finite graph. When S is non-compact with finite area it implies *local* finiteness of $\mathcal{Z}(\phi)$ but not *global*. In this particular case we have the following lemma due to Otal [20, Lemma 6] (the second part is [20, Lemma 1])

Lemma 2.1 *Let S be a hyperbolic surface with finite area and let $\phi : S \to \mathbb{R}$ be a λ-eigenfunction with $\lambda \leq \frac{1}{4}$. Then the closure of $\mathcal{Z}(\phi)$ in $\overline{S}$ is a finite graph without any isolated or free vertex. Moreover, each nodal domain of ϕ has negative Euler characteristic.*

In particular, $\overline{\mathcal{Z}(\phi)}$ is a union of finitely many (not necessarily disjoint) *cycles* in $\overline{S}$ that may contain some of the punctures of S. Next we recall Courant's nodal domain theorem.

Theorem 2.2 *Let S be a closed hyperbolic surface. Then the number of nodal domains of a $\lambda_i(S)$-eigenfunction can be at most $i + 1$.*

The proof (see [7] or [8]) of this theorem works also for finite area hyperbolic surfaces if $\lambda_i < \frac{1}{4}$. In particular, for a hyperbolic surface S with finite area if $\lambda_1(S) < \frac{1}{4}$ then the number of nodal domains of a $\lambda_1(S)$-eigenfunction is at most two. Since any λ_1-eigenfunction ϕ has mean zero, $\mathcal{Z}(\phi)$ must disconnect S. Hence any λ_1-eigenfunction has exactly two nodal domains.

2.3 Cusps

Let S be a finite area hyperbolic surface. Then S is homeomorphic to a closed surface with finitely many points removed. Each of these points, called punctures, has special neighborhoods in S called *cusps*. Denote by ι the parabolic isometry $\iota : z \to z + 2\pi$. For a choice of $t > 0$, a cusp $\mathcal{P}^t$ is the half-infinite cylinder $\{z = x + iy : y > \frac{2\pi}{t}\}/ < \iota >$. The boundary curve $\{y = \frac{2\pi}{t}\}$ is a *horocycle* of length t. The hyperbolic metric on $\mathcal{P}^t$ has the form:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{2.3}$$

Any function $f \in L^2(\mathcal{P}^t)$ has a Fourier development in the x variable of the form

$$f(z) = \sum_{n \in \mathbb{Z}^*} f_n(y) \cos(nx + \theta_n). \tag{2.4}$$

If f satisfy the equation $\Delta f = s(1-s)f$ then the above expression can be simplified as

$$\begin{aligned} f(z) &= f_0(y) + \sum_{j \geq 1} f_j \sqrt{\frac{2jy}{\pi}} K_{s-\frac{1}{2}}(jy) \cos\left(j.x - \theta_j\right) \\ &= f_0(y) + \sum_{j \geq 1} \sqrt{\frac{2jy}{\pi}} K_{s-\frac{1}{2}}(jy) \left(f_j^e \cos(j.x) + f_j^o \sin(j.x)\right) \end{aligned} \tag{2.5}$$

where K_s is the modified *Bessel function* (see [17]) and

$$\begin{aligned} f_0(y) &= f_{0,1} y^s + f_{0,2} y^{1-s} \quad \text{if } s \neq \frac{1}{2} \quad \text{and} \\ f_0(y) &= f_{0,1} y^{\frac{1}{2}} + f_{0,2} y^{\frac{1}{2}} \log y \quad \text{if } s = \frac{1}{2}. \end{aligned} \tag{2.6}$$

The function f is called *cuspidal* if $f_0(y) \equiv 0$. Observe that if $s = \frac{1}{2}$ then, since $f \in L^2(\mathcal{P}^t)$, we must have $f_0(y) \equiv 0$ i.e. f must be cuspidal.

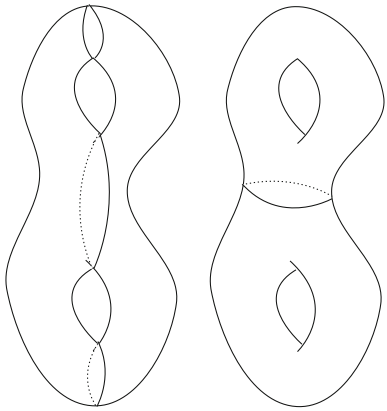
3 Genus two: Proof of Theorem 1.2

We begin by proving that $\mathcal{B}_2(\frac{1}{4})$ disconnects $\mathcal{M}_2$. We argue by contradiction and assume that $\mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ is connected. Now, for any $S \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$: $\lambda_1(S) \leq \frac{1}{4}$ and so $\lambda_1(S)$ is simple by Proposition 1.1. Recall that **if an eigenvalue λ is simple then the nodal set of λ-eigenfunctions is defined without any ambiguity** (see Sect. 2.2). In particular, for any $S \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ the nodal set $\mathcal{Z}_S$ of $\lambda_1(S)$-eigenfunctions is defined without any ambiguity. Now let ϕ_S be a $\lambda_1(S)$-eigenfunction with nodal set $\mathcal{Z}(\phi_S) = \mathcal{Z}_S$. Since ϕ_S is an eigenfunction corresponding to $\lambda_1(S)$, by Courant's nodal domain theorem, $S \backslash \mathcal{Z}(\phi_S)$ has exactly two components. Denote by S^+ (resp. S^-) the component of $S \backslash \mathcal{Z}(\phi_S)$ where ϕ_S is positive (resp. negative). By the Euler–Poincaré formula (2.2) applied to the graph $\mathcal{Z}(\phi_S)$ we have the following equality:

$$\chi(S) = \chi(S^+) + \chi(S^-) + \chi(\mathcal{Z}(\phi_S)). \tag{3.1}$$

Since $\chi(S) = -2$ and both $\chi(S^+)$ and $\chi(S^-)$ are negative by Lemma 2.1 we conclude from (3.1) that $\chi(\mathcal{Z}(\phi_S)) = 0$. This immediately implies that $\mathcal{Z}(\phi_S)$ consists of disjoint

Fig. 1 Decomposition



simple closed curve(s) that divide S into exactly two components. From Theorem 2.1 it is clear that each curve in $\mathcal{Z}(\phi_S)$ appear in the boundary of each of S^+ and S^-. This, together with the simplicity of $\mathcal{Z}(\phi_S)$, implies that the number of boundary components of S^+ and S^- are the same. Now a simple Euler characteristic counting provides the following description (Fig. 1).

Lemma 3.1 *For any $S \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ the nodal set $\mathcal{Z}_S$ consists either of three smooth simple closed curves that divide S into two pair of pants (the first picture below) or of a unique smooth simple closed curve that divides S into two tori with one hole (the second picture below).*

Now we have the following:

Lemma 3.2 *Let $S \in \mathcal{M}_2$ be such that $\lambda_1(S)$ is simple and the nodal set $\mathcal{Z}_S$ of $\lambda_1(S)$-eigenfunctions is also simple. Then S has a neighborhood $\mathcal{N}(S)$ in $\mathcal{M}_2$ such that $\lambda_1(S')$ is simple for any $S' \in \mathcal{N}(S)$ and the nodal set $\mathcal{Z}_{S'}$ of $\lambda_1(S')$-eigenfunctions isotopic to $\mathcal{Z}_S$.*

Proof First observe that since $\lambda_1(S)$ is simple, by the continuity of λ_1 as a function, we have a neighborhood $\mathcal{N}'(S)$ of S in $\mathcal{M}_2$ such that $\lambda_1(S')$ is simple for any $S' \in \mathcal{N}'(S)$. Let ϕ_S be a $\lambda_1(S)$-eigenfunction and let S^+ and S^- be the two components of $S \backslash \mathcal{Z}(\phi_S)$ such that ϕ_S has positive sign on S^+ and negative sign on S^-. Consider a tubular neighborhood $\mathcal{T}_S$ of $\mathcal{Z}(\phi_S)$ in S. By [19, Theorem 3.36] (see also [12,16]) we have a neighborhood $\mathcal{N}(S) \subset \mathcal{N}'(S)$ of S such that for any $S' \in \mathcal{N}(S)$, one can obtain a $\lambda_1(S')$-eigenfunction $\phi_{S'}$ that has positive sign on $S^+ \backslash \mathcal{T}_S$ and negative sign on $S^- \backslash \mathcal{T}_S$. In particular, $\mathcal{Z}_{S'} = \mathcal{Z}(\phi_{S'}) \subset \mathcal{T}_S$. Hence by the description of $\mathcal{Z}_{S'}$ as in Lemma 3.1 the proof follows.

Therefore, there exists $S \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$ such that $\mathcal{Z}_S$ consists of only one curve if and only if for all $S' \in \mathcal{M}_2 \backslash \mathcal{B}_2(\frac{1}{4})$, $\mathcal{Z}_{S'}$ consists of only one curve. This is a contradiction to our next result Proposition 3.1.

Definition 3.1 The *systole* $s(S)$ of a surface S is the minimum of the lengths of closed geodesics on S. The *injectivity radius* of S at a point p is the radius of the largest geodesic disc that can be embedded in S with center p. For any $\epsilon > 0$ the set of points of S with injectivity radius at least ϵ is denoted by $S^{[\epsilon,\infty)}$. Each point in the complement $S^{(0,\epsilon)} = S \backslash S^{[\epsilon,\infty)}$ has injectivity radius at most ϵ. $S^{[\epsilon,\infty)}$ and $S^{(0,\epsilon)}$ are respectively called *ϵ-thick part* and *ϵ-thin part* of S.

Proposition 3.1 *Let S be a finite area hyperbolic surface of type (g, n). Let $G = (\gamma_i)_{i=1}^k$ be a collection of smooth, mutually non-intersecting simple closed curves on S that separates S in exactly two components. Assume that G is minimal in the sense that no proper subset of G can separate S. Then given any $\epsilon, \delta > 0$ there exists a finite area hyperbolic surface S_G of type (g, n) with $s(S_G) < \epsilon$ such that $\lambda_1(S_G) < \delta$ is simple and the nodal set $\mathcal{Z}_{S_G}$ of $\lambda_1(S_G)$-eigenfunctions is isotopic to G.*

Remark 3.1 For particular cases it is not very difficult to construct two collections of curves on S, as in the above lemma, that are not isotopic. In the case $(g, n) = (2, 0)$ claim 3.1 provides two such collections. Therefore Proposition 3.1 indeed provide two surfaces S_1, $S_2 \in \mathcal{M}_2\backslash\mathcal{B}_2(\frac{1}{4})$ such that $\mathcal{Z}_{S_1}$ is not isotopic to $\mathcal{Z}_{S_2}$.

The proof of Proposition 3.1 uses the behavior of sequences of small eigenpairs over degenerating sequences of hyperbolic surfaces. For precise definitions of these concepts we refer the reader to [19]. We immediately remark that such behavior has been widely studied in the literature, see for example [9,12,16,24]. However, the terminology used in the next proof follows those in [19].

Proof Without loss of generality we may assume that each curve in G is a geodesic. Extend G to a pants decomposition $P = (\gamma_i)_{i=1}^{3g-3+n}$ of S [4, p-94]. Let (l_i, θ_i) denote the Fenchel–Nielsen coordinates on $\mathcal{T}_{g,n}$ with respect to $(\gamma_i)_{i=1}^{3g-3+n}$. Here l_i denotes the length parameter and θ_i denotes the twist parameter along γ_i.

Now consider the sequence of surfaces (S_m) in $\mathcal{T}_{g,n}$ such that $l_i(S_m) = \frac{1}{m}$ for $i \leq k$, $l_j = c_1 > 0$ for $j > k$ and $\theta_j = c_2 > 0$ for $1 \leq j \leq 3g - 3 + n$. Then, up to extracting a subsequence, (S_m) converges to a finite area hyperbolic surface $S_\infty \in \partial\mathcal{M}_{g,n}$. Let us denote the extracted subsequence by (S_m) itself. Observe that S_∞ is obtained from S by pinching the geodesics in G. Namely, for each $i = 1, \ldots, k$ there is a geodesic γ_i^m in S_m, in the homotopy class of γ_i, whose length tends to zero as $m \to \infty$.

The number of components of $S_\infty \in \overline{\mathcal{M}_{g,n}}$ is exactly two. Hence by Colbois and Courtois [9], $\lambda_1(S_m) \to 0$ and all other eigenvalues of S_m stay away from zero. In particular $\lambda_1(S_m)$ is simple for m sufficiently large. Let ϕ_{S_m} be a $\lambda_1(S_m)$-eigenfunction with L^2-norm 1. Recall that we want to prove that for any $\epsilon, \delta > 0$ there exists a S_G with $s(S_G) < \epsilon$ such that $\lambda_1(S_G) < \delta$ is simple and the nodal set $\mathcal{Z}_{S_G}$ of any $\lambda_1(S_G)$-eigenfunction is isotopic to G. Since $s(S_m) \to 0$ by construction and $\lambda_1(S_m) \to 0$ by above it suffices to prove that $\mathcal{Z}(\phi_{S_m})$ is isotopic to G for sufficiently large m.

Now we apply [19, Theorem 3.34] to extract a subsequence of ϕ_{S_m} that converges uniformly over compacta to a 0-eigenfunction ϕ_∞ of S_∞ with L^2-norm 1. Let us denote the extracted subsequence by (S_m) itself. Since 0-eigenfunctions are constant functions, ϕ_∞ is constant on each components of S_∞.

Lemma 3.3 *The two constant values of ϕ_∞ on the two components of S_∞ are non-zero and have opposite sign.*

Proof For $\epsilon > 0$ let us denote the L^2-norm of ϕ_{S_m} restricted to $S_m^{(0,\epsilon)}$ by $\|\phi_{S_m}\|_{S_m^{(0,\epsilon)}}$. By the uniform convergence of ϕ_{S_m} to ϕ_∞ over compacta we have

$$\int_{S_\infty^{[\epsilon,\infty)}} \phi_\infty^2 = \lim_{m\to\infty} \int_{S_m^{[\epsilon,\infty)}} \phi_{S_m}^2 = 1 - \lim_{m\to\infty} \|\phi_{S_m}\|^2_{S_m^{(0,\epsilon)}}.$$

Since $\int_{S_\infty}\phi_\infty^2 = \lim_{\epsilon\to 0}\int_{S_\infty^{[\epsilon,\infty)}}\phi_\infty^2 = 1$ we obtain that for any $\delta > 0$ there exists $\epsilon > 0$ such that $\lim_{m\to\infty}\|\phi_{S_m}\|_{S_m^{(0,\epsilon)}} \leq \delta$. Now

$$\left|\int_{S_\infty^{[\epsilon,\infty)}}\phi_\infty\right| = \lim_{m\to\infty}\left|\int_{S_m^{[\epsilon,\infty)}}\phi_{S_m}\right| = \left|0 - \lim_{m\to\infty}\int_{S_m^{(0,\epsilon)}}\phi_{S_m}\right|$$
$$\leq \lim_{m\to\infty}\sqrt{|S_m^{(0,\epsilon)}|}\|\phi_{S_m}\|_{S_m^{(0,\epsilon)}} \text{(by Holder inequality)} \leq \delta \lim_{m\to\infty}\sqrt{|S_m^{(0,\epsilon)}|}.$$

Here $|S_m^{(0,\epsilon)}|$ denotes the area of $S_m^{(0,\epsilon)}$. Recall that, for any $m \in \mathbb{N}\cup\infty$, $\lim_{\epsilon\to 0}|S_m^{(0,\epsilon)}| = 0$. So for $m \geq 1$ and ϵ sufficiently small:

$$\left|\int_{S_\infty^{[\epsilon,\infty)}}\phi_\infty\right| < \delta \quad\text{and}\quad |S_m^{(0,\epsilon)}| < \delta.$$

Finally, taking ϵ to be sufficiently small, we calculate:

$$\left|\int_{S_\infty}\phi_\infty\right| \leq \left|\int_{S_\infty^{[\epsilon,\infty)}}\phi_\infty\right| + \left|\int_{S_\infty^{(0,\epsilon)}}\phi_\infty\right| \leq \delta + \sqrt{|S_\infty^{(0,\epsilon)}|}\|\phi_{S_\infty}\|_{S_\infty^{(0,\epsilon)}} \leq 2\delta$$

since $\|\phi_{S_\infty}\|_{S_\infty^{(0,\epsilon)}} < \|\phi_{S_\infty}\| = 1$. Since δ is arbitrary we conclude that $\int_{S_\infty}\phi_\infty = 0$. Hence ϕ_∞ has L^2-norm 1 and mean zero.

Since ϕ_∞ has L^2-norm 1 at least one of the two constant values of ϕ_∞ on the two components of S_∞ is non-zero. Since ϕ_∞ has mean zero both of these values are non-zero with opposite sign.

As the length of γ_i^m tends to zero, we may assume that the collar neighborhood C_i^m of γ_i^m with two boundary components of length 1 embeds in S_m and $(C_i^m)_{i=1}^k$ are mutually disjoint. At this point we recall that G is minimal in the sense that no proper subset of G can separate S. Hence not only $S_m\backslash\cup_{i=1}^k(C_i^m)$ separates S in exactly two components but also no proper sub-collection of $(C_i^m)_{i=1}^k$ can separate S_m. In particular, for each i, the limits of the two components of ∂C_i^m belong to the two different components of S_∞. Using Lemma 3.3 let us denote the limits of these two boundary sets by $B_i^\infty(+)$ and $B_i^\infty(-)$ such that $\phi_\infty|_{B_i^\infty(+)} > 0$ and $\phi_\infty|_{B_i^\infty(-)} < 0$. Correspondingly denote the two components of ∂C_i^m by $B_i^m(+)$ and $B_i^m(-)$ such that $B_i^\infty(\pm)$ is the limit of $B_i^m(\pm)$ respectively. By the uniform convergence of ϕ_{S_m} to ϕ_∞ over compacta we conclude that, for sufficiently large m, $\phi_{S_m}|_{B_i^m(+)} > 0$ and $\phi_{S_m}|_{B_i^m(-)} < 0$. Hence, for m sufficiently large, at least one component of $\mathcal{Z}(\phi_{S_m})$ is contained in C_i^m.

Let Z_i denote the union of the components of $\mathcal{Z}(\phi_{S_m})$ that are contained in C_i^m. Let α be a simple closed loop in Z_i. Since $\pi_1(C_i^m)$ is $\mathbb{Z}$ there are only two possibilities for α. Either it bounds a disc in C_i^m or it is homotopic to γ_i^m. Since $\lambda_1(S_m)$ is small, each component of $S_m\backslash\mathcal{Z}(\phi_{S_m})$ has negative Euler characteristic by Lemma 2.1. This discards the possibility that α bounds a disc in C_i^m. Hence α is homotopic to γ_i^m. Let β be another simple closed loop in Z_i. Then β is also homotopic to γ_i^m implying that one of the components of $S_m\backslash\mathcal{Z}(\phi_{S_m})$ has non-negative Euler characteristic. This leaves us with the observation that each C_i^m contains exactly one loop α_i^m from $\mathcal{Z}(\phi_{S_m})$. By remark 2.1 α_i^m is in fact smooth. Therefore we have an isotopy of S that sends α_i^m to γ_i^m. Combining these isotopies we obtain that $\mathcal{Z}(\phi_{S_m})$ is isotopic to $(\gamma_i^m)_{i=1}^k$.

It remains to show that $\mathcal{B}_2(\frac{1}{4})$ is unbounded. We argue by contradiction and assume that $\mathcal{B}_2(\frac{1}{4})$ is bounded. Then we have $\epsilon > 0$ such that $\mathcal{B}_2(\frac{1}{4})$ is contained in the compact set $\mathcal{I}_\epsilon = \{S \in \mathcal{M}_2 : s(S) \geq \epsilon\}$ [1]. Now applying Proposition 3.1 obtain S_1 and S_2 in $\mathcal{M}_2$ such that $s(S_i) < \epsilon$, $\lambda_1(S_i) < \frac{1}{4}$ is simple and the nodal set $\mathcal{Z}_{S_1}$ of $\lambda_1(S_1)$-eigenfunctions is not isotopic to the nodal set $\mathcal{Z}_{S_2}$ of $\lambda_1(S_2)$-eigenfunctions. On the other hand, since $\mathcal{M}_2\backslash\mathcal{I}_\epsilon$ is path connected (see Lemma 5.3) we may have a path β in $\mathcal{M}_2\backslash\mathcal{I}_\epsilon \subset \mathcal{M}_2\backslash\mathcal{B}_2(\frac{1}{4})$ that joins S_1 and S_2. By the last inclusion $\beta \subset \mathcal{M}_2\backslash\mathcal{B}_2(\frac{1}{4})$ we get that λ_1 is simple along β and we may apply Lemma 3.2 to obtain that the nodal set of λ_1-eigenfunctions is constant, up to isotopy, along β. In particular, $\mathcal{Z}_{S_1}$ is isotopic to $\mathcal{Z}_{S_2}$, a contradiction.

3.1 Proof of Theorem 1.3

The case $(g, n) = (2, 0)$ is the content of the above theorem. It remains to prove Theorem 1.3 for $(g, n) = (1, 2)$ and $(0, 4)$. **For the rest of the proof we refer to the pair (g, n) for only these two cases.** We argue by contradiction and assume that $\mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$ is connected. By definition $\lambda_1(S) < \frac{1}{4}$ for any $S \in \mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$. Hence $\lambda_1(S)$ is an eigenvalue and by [21] it is the only non-zero small eigenvalue of S. Hence the nodal set $\mathcal{Z}_S$ of $\lambda_1(S)$-eigenfunctions is defined without any ambiguity. Let ϕ_S be a $\lambda_1(S)$-eigenfunction with nodal set $\mathcal{Z}(\phi_S) = \mathcal{Z}_S$. Denote by $\overline{S}$ the surface obtained from S by filling in its punctures and by $\overline{\mathcal{Z}(\phi_S)}$ the closure of $\mathcal{Z}(\phi_S)$ in $\overline{S}$. By Lemma 2.1 $\overline{\mathcal{Z}(\phi_S)}$ is a finite graph without any isolated or free vertex. Now the Euler–Poincaré formula (2.2) applied to the graph $\overline{\mathcal{Z}(\phi_S)}$ provides the equality

$$\chi(\overline{S}) - k = \chi\left(\overline{S}\backslash\overline{\mathcal{Z}(\phi_S)}\right) + \chi\left(\overline{\mathcal{Z}(\phi_S)}\right) \tag{3.2}$$

where k is the number of punctures of S that do not lie on $\overline{\mathcal{Z}(\phi_S)}$. By Lemma 2.1 each component of $\overline{S}\backslash\overline{\mathcal{Z}(\phi_S)}$ has negative Euler characteristic and so $\chi(\overline{S}\backslash\overline{\mathcal{Z}(\phi_S)}) \leq -2$. Recall that the Euler characteristic of a finite graph without any isolated or free vertex is always ≤ 0. Now, for $(g, n) = (1, 2)$, $\chi(\overline{S}) = 0$ and so we have the only possibility $k = 2$ and $\chi(\overline{\mathcal{Z}(\phi_S)}) = 0$. Also, for $(g, n) = (0, 4)$, $\chi(\overline{S}) = 2$ leaves us with the only possibility $k = 4$ and $\chi(\overline{\mathcal{Z}(\phi_S)}) = 0$. Hence, in both cases, none of the punctures of S lie on $\overline{\mathcal{Z}(\phi_S)}$. In particular, $\overline{\mathcal{Z}(\phi_S)} = \mathcal{Z}(\phi_S)$ is a compact subset of S. Since $\chi(\overline{\mathcal{Z}(\phi_S)}) = 0$ we conclude that $\mathcal{Z}(\phi_S)$ is a union of simple closed curves in S. Following arguments similar to those in the genus two case we obtain the following description.

Lemma 3.4 *Let $S \in \mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$.*

(i) *If $(g, n) = (1, 2)$ then $\mathcal{Z}_S = \mathcal{Z}(\phi_S)$ consists of either exactly one simple closed curve or two simple closed curves. In the first case $\mathcal{Z}_S$ divides S into two components one of which is a surface of genus one with a copy of $\mathcal{Z}_S$ as its boundary and the other one is a twice punctured sphere with a copy of $\mathcal{Z}_S$ as its boundary. In the last case $\mathcal{Z}_S$ divides S into two components each of which is a once punctured sphere with two boundary components coming from $\mathcal{Z}_S$.*

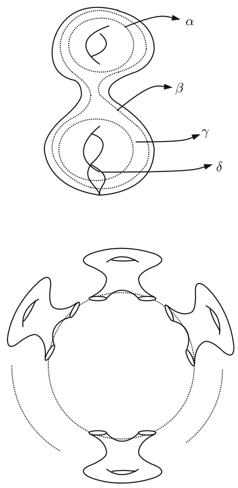
(ii) *If $(g, n) = (0, 4)$ then $\mathcal{Z}_S$ consists of exactly one simple closed curve (there are two possibilities for this up to isotopy) that separates S into two components each of which is a twice punctured sphere with one boundary component coming from $\mathcal{Z}_S$.*

Next we have the following modified version of Lemma 3.2. Let $S \in \mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$ with nodal set $\mathcal{Z}_S$ of $\lambda_1(S)$-eigenfunctions.

Lemma 3.5 *There exists a neighborhood $\mathcal{N}(S)$ of S in $\mathcal{M}_{g,n}$ such that $\lambda_1(S')$ is simple for any $S' \in \mathcal{N}(S)$ and the nodal set $\mathcal{Z}_{S'}$ is isotopic to $\mathcal{Z}_S$.*

Proof By assumption $\lambda_1(S) < \frac{1}{4}$ and so λ_1 defines a continuous function in a neighborhood of S by [12](see also [9,19]). Hence we have a neighborhood $\mathcal{N}'(S) \subset \mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$ of S.

Fig. 2 Cover



In particular, for $S' \in \mathcal{N}'(S)$, the nodal set $\mathcal{Z}_{S'}$ of $\lambda_1(S')$-eigenfunctions has the description in Lemma 3.4. Let ϕ_S be a $\lambda_1(S)$-eigenfunction. Now consider a tubular neighborhood $\mathcal{T}_S$ of $\mathcal{Z}_S$ in S such that $S\backslash\mathcal{T}_S$ has two components S^+ and S^- with $\phi_S|_{S^+} > 0$ and $\phi_S|_{S^-} < 0$. Furthermore, using Lemma 3.4 we assume that the boundary components $\partial S^{\pm}$of $S^{\pm}$ are disjoint union of simple closed curves.

Now, as λ_1 is simple and $< \frac{1}{4}$ on $\mathcal{N}'(S)$, by [12], for any compact subset K of S, one can find $\lambda_1(S')$-eigenfunctions $\phi_{S'}$ such that the map $\Phi : K \times \mathcal{N}'(S) \to \mathbb{R}$ given by $\Phi(x, S') = \phi_{S'}(x)$ is continuous. Considering $K = \partial S^+ \cup \partial S^-$ we obtain a neighborhood $\mathcal{N}(S) \subset \mathcal{N}'(S)$ of S such that for any $S' \in \mathcal{N}(S)$: $\phi_{S'}|_{\partial S^+} > 0$ and $\phi_{S'}|_{\partial S^-} < 0$. In particular, $\mathcal{Z}(\phi_{S'}) \subset \mathcal{T}_S$ for any $S' \in \mathcal{N}(S)$. Finally by the description of $\mathcal{Z}_{S'} = \mathcal{Z}(\phi_{S'})$ in Lemma 3.4 we obtain the lemma.

Continuation of proof of Theorem 1.3 Since by our assumption $\mathcal{M}_{g,n}\backslash\mathcal{C}_{g,n}(\frac{1}{4})$ is connected the above claim implies that only one of the two possibilities in Lemma 3.4 can actually occur. This is a contradiction to Proposition 3.1.

Now we show that $\mathcal{C}_{1,2}(\frac{1}{4})$ is unbounded. We argue by contradiction and assume that $\mathcal{C}_{1,2}(\frac{1}{4})$ is bounded. Then we have $\epsilon > 0$ such that $\mathcal{C}_{1,2}(\frac{1}{4})$ is contained in the compact set $\mathcal{I}_\epsilon = \{S \in \mathcal{M}_{1,2} : s(S) \geq \epsilon\}$ [1]. Applying Lemma 3.1 we obtain S_1 and S_2 in $\mathcal{M}_{1,2}$ such that $s(S_i) < \epsilon$, $\lambda_1(S_i) < \frac{1}{4}$ is simple and the nodal set $\mathcal{Z}_{S_1}$ of $\lambda_1(S_1)$-eigenfunctions is not isotopic to the nodal set $\mathcal{Z}_{S_2}$ of $\lambda_1(S_2)$-eigenfunctions. On the other hand, since $\mathcal{M}_{1,2}\backslash\mathcal{I}_\epsilon$ is path connected (see Lemma 5.3) we may have a path β in $\mathcal{M}_{1,2}\backslash\mathcal{I}_\epsilon \subset \mathcal{M}_{1,2}\backslash\mathcal{C}_{1,2}(\frac{1}{4})$ that joins S_1 and S_2. By the last inclusion $\beta \subset \mathcal{M}_2\backslash\mathcal{C}_{1,2}(\frac{1}{4})$ we get that λ_1 is simple along β and we may apply Lemma 3.2 to obtain that $\mathcal{Z}_{S_1}$ is isotopic to $\mathcal{Z}_{S_2}$, a contradiction (Fig. 2).

4 Branches of eigenvalues

In this section we consider branches of eigenvalues along paths in $\mathcal{T}_g$. Main purpose of doing so is that the multiplicity of λ_i, in particular λ_1, is not one in general (see Theorem 1.6). Therefore along 'nice' paths in $\mathcal{T}_g$ the functions λ_i may not be 'nice' enough (see Sect. 1.1). However, Theorem 1.4 shows that up to certain choice at points of multiplicity λ_i's are in fact 'nice'. This 'nice' choice makes λ_i into a branch of eigenvalues (see Sect. 1.1). Theorem 1.5 says that if we restrict ourselves to branches of eigenvalues then we have a positive answer to Conjecture 1.2, namely there are branches of eigenvalues that start as λ_1 and becomes more than $\frac{1}{4}$.

Proof (Proof of Theorem 1.5) We begin by explaining the embedding $\Pi : \mathcal{T}_2 \to \mathcal{T}_g$ (see the next figure cover). Let S be the closed hyperbolic surface of genus two and $\alpha, \beta, \gamma, \delta$ are four geodesics on S as in the picture below. Now cut S along δ to obtain a hyperbolic surface S^* with genus one and two geodesic boundaries (each a copy of δ). Consider $g-1$ many copies of S^* and glue them along their consecutive boundaries after arranging them along a circle as in the picture below. Let $\Pi(S)$ denote the resulting hyperbolic surface.

Now take a geodesic pants decomposition $(\xi_i)_{i=1,2,3}$ of S involving $\delta = \xi_3$ and consider the Fenchel–Nielsen coordinates $(l_i, \theta_i)_{i=1,2,3}$ on $\mathcal{T}_2$ with respect to this pants decomposition. Here $l_i = l(\xi_i)$ is the length of the closed geodesic ξ_i and θ_i is the twist parameter at ξ_i. The images of $(\xi_i)_{i=1,2,3}$ in $\Pi(S)$, $(\xi_i^j)_{i=1,2,3;\, j=1,2,\ldots,g-1}$ is a geodesic pants decomposition of $\Pi(S)$. Consider the the Fenchel–Nielsen coordinates $(l_i^j, \theta_i^j)_{i=1,2,3;\, j=1,2,\ldots,g-1}$ on $\mathcal{T}_g$ with respect to this pants decomposition. As before, $l_i^j = l(\xi_i^j)$ is the length of the closed geodesic ξ_i^j and θ_i^j is the twist parameter at ξ_i^j. With respect to these pants decompositions Π is expressed as

$$(l_1, l_2, l_3, \theta_1, \theta_2, \theta_3) \to (\underbrace{l_1, l_2, l_3, \theta_1, \theta_2, \theta_3}_{1}, \ldots, \underbrace{l_1, l_2, l_3, \theta_1, \theta_2, \theta_3}_{g-1}). \tag{4.1}$$

This is an analytic map and the image $\Pi(S)$ of any $S \in \mathcal{T}_2$ has an isometry τ of order $(g-1)$ that sends one 6-tuple $(l_1, l_2, l_3, \theta_1, \theta_2, \theta_3)$ to the next one. Also $\Pi(S)/\tau$ is isometric to S i.e. $\Pi(S)$ is a $(g-1)$ sheeted covering of S. Hence each eigenvalue of S is also an eigenvalue of $\Pi(S)$. In particular, a branch λ_t of eigenvalues in $\mathcal{T}_2$ along $\eta(t)$ is a branch of eigenvalues in $\mathcal{T}_g$ along $\Pi(\eta(t))$.

To finish the proof we need only to find $S \in \mathcal{T}_2$ such that $\lambda_1(S) = \lambda_1(\Pi(S))$. Once we find such a S, we can consider any analytic path η in $\mathcal{T}_2$ such that $\eta(o) = S$ and $\lambda_1(\eta(1)) > \frac{1}{4}$. Then the branch of eigenvalues $\lambda_t = \lambda_1(\eta(t))$ along $\Pi(\eta(t))$ would be a branch that we seek.

To show this we employ the technique in Proposition 3.1. Let S_n be a sequence of surfaces of genus two on which the lengths of the geodesics α, β and γ tends to zero. In particular, $S_n \to S_\infty \in \mathcal{M}_{0,3} \cup \mathcal{M}_{0,3}$ implying $\lambda_1(S_n) \to 0$ and $\lambda_2(S_n) \nrightarrow 0$. The sequence $\Pi(S_n)$ converges to a surface in $\mathcal{M}_{0,g+1} \cup \mathcal{M}_{0,g+1}$ and so $\lambda_1(\Pi(S_n)) \to 0$ and $\lambda_2(\Pi(S_n)) \nrightarrow 0$. So for large n, $\lambda_1(S_n) < \lambda_2(\Pi(S_n))$ implying $\lambda_1(S_n) = \lambda_1(\Pi(S_n))$.

5 Punctured spheres

We begin this section by recapitulating the ideas in [5]. By purely number theoretic methods Atle Selberg showed that for any congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$, $\lambda_1(\mathbb{H}/\Gamma) \geq \frac{3}{16}$. The purpose in [5] was to construct explicit closed hyperbolic surfaces with λ_1 close to $\frac{3}{16}$. To achieve this goal the authors of [5] considered principal congruence subgroups Γ_n (see introduction) and corresponding finite area hyperbolic surfaces $\mathbb{H}/\Gamma_n$. Then they replaced the cusps in $\mathbb{H}/\Gamma_n$, which is even in number, by closed geodesics of small length t and glued them in pairs (see [5] for details). The surface S_t obtained in this way is closed, their genus g is independent of t and as $t \to 0$, $S_t \to \mathbb{H}/\Gamma_n$ in the compactification of the moduli space

$\mathcal{M}_g$. Rest of the proof showed that λ_1 is lower semi-continuous over the family S_t. A novel modification of this approach in [6] together with the result of Kim and Sarnak provides Theorem 1.1.

Limiting properties of eigenvalues over degenerating family of hyperbolic metrics have been studied well in the literature (to name a few Hejhal [12], Colbois and Courtois [9], Ji [16], Wolpert [24], Judge [17]) (see also [19, Theorem 2]). These limiting results can be summarized as:

Theorem 5.1 *Let (S_m) be a sequence of hyperbolic surfaces in $\mathcal{M}_{g,n}$ that converges to a finite area hyperbolic surface $S \in \partial\mathcal{M}_{g,n}$. Let (λ_m, ϕ_m) be an eigenpair of S_m such that $\lambda_m \to \lambda < \infty$. Then, up to extracting a subsequence and up to rescaling, the sequence (ϕ_m) converges to a generalized eigenfunction, uniformly over compacta, if one of the following is true (i) $n = 0$ ([16]) (ii) $n \neq 0$ and $\lambda < \frac{1}{4}$ ([9,12]) (iii) $n \neq 0$ and $\lambda > \frac{1}{4}$ ([24]) (iii) $n \neq 0$, $\lambda_m \leq \frac{1}{4}$ and ϕ_m is cuspidal ([19]).*

Recall that there is a copy of $\mathcal{M}_{0,2g+n}$ in the compactification $\overline{\mathcal{M}_{g,n}}$ of $\mathcal{M}_{g,n}$. The ideas in [5] along with above limiting results imply the following.

Lemma 5.1 *For any pair (g, n), $\Lambda_1(g, n) \geq \Lambda_1(0, 2g + n)$.*

Motivated by this we focus on $\Lambda_1(0, n)$. Although we would not be able to prove Conjecture 1.2 we have Theorem 1.7 on the multiplicity of λ_1 which we prove now.

5.1 Proof of Theorem 1.7

Let S be a finite area hyperbolic surface of genus 0 and assume that $\lambda_1(S)$ is a small eigenvalue. Following the discussion in Sect. 1.2 $\lambda_1(S) < \frac{1}{4}$. Let $\overline{S}$ denote the closed surface obtained by filling in the punctures of S. Let ϕ be a $\lambda_1(S)$-eigenfunction. Then the closure $\overline{\mathcal{Z}(\phi)}$ of the nodal set $\mathcal{Z}(\phi)$ of ϕ is a finite graph in $\overline{S}$ by Lemma 2.1. In particular, $\overline{\mathcal{Z}(\phi)}$ is a union of closed loops (not necessarily disjoint) in $\overline{S}$. Observe also that the number of components of $\overline{S}\backslash\overline{\mathcal{Z}(\phi)}$ is same as that of $S\backslash\mathcal{Z}(\phi)$.

Now let $\overline{\mathcal{Z}(\phi)}$ consists of more than one closed loop. Then by Jordan curve theorem the number of components of $\overline{S}\backslash\overline{\mathcal{Z}(\phi)}$ is at least three. This is a contradiction to Courant's nodal domain Theorem 2.2 which says that a $\lambda_1(S)$-eigenfunction can have at most two nodal domains. Hence we conclude that $\overline{\mathcal{Z}(\phi)}$ consists of exactly one closed loop. In particular, we have the following description of $\mathcal{Z}(\phi)$ at any puncture.

Lemma 5.2 *If one of the punctures p of S is in $\overline{\mathcal{Z}(\phi)}$ then the number of arcs in $\overline{\mathcal{Z}(\phi)}$ emanating from p is at most two.*

Let $\lambda_1(S) = s(1 - s)$ with $s \in (\frac{1}{2}, 1]$. Let p be one of the punctures of S. Let $\mathcal{P}^t$ be a cusp around p (see Sect. 2.3). Recall that S being a punctured sphere, does not have any small cuspidal eigenvalue [13,20]. Thus any $\lambda_1(S)$-eigenfunction ϕ is a linear combination of residues of Eisenstein series (see [14]). It follows from [14, Thorem 6.9] that the y^s term can not occur in the Fourier development (see (2.5) and (2.6)) of these residues in $\mathcal{P}^t$. Hence ϕ has a Fourier development in $\mathcal{P}^t$ of the form:

$$\phi(x, y) = \phi_0 y^{1-s} + \sum_{j\geq 1}\sqrt{\frac{2jy}{\pi}}K_{s-\frac{1}{2}}(jy)(\phi_j^e\cos(j.x) + \phi_j^o\sin(j.x)). \tag{5.1}$$

Now we consider the space $\mathcal{E}_1$ generated by $\lambda_1(S)$-eigenfunctions. The map $\pi : \mathcal{E}_1 \to \mathbb{R}^3$ given by $\pi(\phi) = (\phi_0, \phi_1^e, \phi_1^o)$ is linear and so if $\dim\mathcal{E}_1 > 3$ then $\ker\pi$ is non-empty. Let

$\psi \in \ker \pi$ i.e. $\psi_0 = \psi_1^e = \psi_1^o = 0$. Then by the result [17] of Judge, the number of arcs in $\mathcal{Z}(\psi)$ emanating from p is at least four, a contradiction to Lemma 5.2.

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Appendix

For the convenience of the reader we give a proof of the fact that, for $(g, n) \neq (0, 4), (1, 1)$, the complement $\mathcal{M}_{g,n} \backslash \mathcal{I}_\epsilon$ of the compact set $\mathcal{I}_\epsilon = \{S \in \mathcal{M}_{g,n} : s(S) \geq \epsilon\}$ [1] is path connected.

Lemma 5.3 *For any* $(g, n) \neq (0, 4), (1, 1)$ *with* $2g - 2 + n > 0$ *and any* $\epsilon > 0$ *the set* $\mathcal{M}_{g,n} \backslash \mathcal{I}_\epsilon$ *is path connected.*

Proof Let S_1 and S_2 be two surfaces in $\mathcal{M}_{g,n}$ such that $s(S_i) < \epsilon$. So we have simple closed geodesics γ_1 on S_1 and γ_2 on S_2 such that the length l_{γ_i} of γ_i is $< \epsilon$. Recall that it has always been our practise to treat $\mathcal{M}_{g,n}$ as a subset of all possible metrics on a fixed surface S and the geodesics are understood to be parametric curves on S that satisfy certain differential equations provided by the metric.

With this understanding let us first assume that γ_1 does not intersect γ_2. So we may consider a pants decomposition P of S containing both γ_1 and γ_2. Let the Fenchel–Nielsen coordinates of S_i be given by $(l_j(S_i), \theta_j(S_i))_{j=1}^{3g-3+n}$. Here l_1, l_2 are the length parameters along γ_1, γ_2 and θ_1, θ_2 are twist parameters along γ_1, γ_2. Then consider the path $\beta : [0, 1] \to \mathcal{T}_2$ given by:

$$l_1(\beta(t)) = \begin{cases} l_1(S_1) & \text{if } t \in [0, \frac{1}{2}], \\ 2(1-t)l_1(S_1) + (2t-1)l_1(S_2) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

$$l_2(\beta(t)) = \begin{cases} (1-2t)l_2(S_1) + 2tl_2(S_2) & \text{if } t \in [0, \frac{1}{2}], \\ l_2(S_2) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

$l_3(\beta(t)) = (1-t)l_3(S_1) + tl_3(S_2)$ and $\theta_j(\beta(t)) = (1-t)\theta_j(S_1) + t\theta_j(S_2)$. Since $l_1(\beta(t)) < \epsilon$ for $t \in [0, \frac{1}{2}]$ and $l_2(\beta(t)) < \epsilon$ for $t \in [\frac{1}{2}, 1]$ we observe that $s(\beta(t)) < \epsilon$ for all t. The image of β under the quotient map $\mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ produces the required path joining S_1 and S_2.

Now let us assume that γ_1 intersects γ_2. Let γ be a simple closed geodesic that does not intersect γ_1 and γ_2. By our assumption i.e. $(g, n) \neq (0, 4), (1, 1)$ such a geodesic exists. Then by the procedure described above both S_1 and S_2 can be joined by a path in $\mathcal{M}_{g,n} \backslash \mathcal{I}_\epsilon$ to a surface on which γ has length $< \epsilon$. This finishes the proof.

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